Parametrized Surfaces

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Parametrized Surfaces

We have defined **curves in the plane** in three different ways :

- Explicit form : $y = f(x)$
- Implicit form : $F(x, y) = 0$
- Parametric vector form : $\mathrm{r}(t) = f(t)\mathrm{i} + g(t)\mathrm{j}, \quad a \leq t \leq q$

Parametrized Surfaces

We have analogous definitions of surfaces in space:

- Explicit form : $z = f(x, y)$
- Implicit form : $F(x, y, z) = 0$

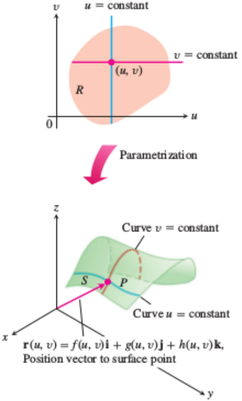
There is also a parametric form that gives the position of a point on the surface as a vector funtion of two variables.

We now extend the investigation of surface area and surface integrals to surfaces described parametrically.

Let

$$\mathrm{r}(u,v) = f(u,v)\mathrm{i} + g(u,v)\mathrm{j} + h(u,v)\mathrm{k} \tag{1}$$

be a continuous vector function that is defined on a region R in the uv-plane and one-to-one on the interior of R.



Parametrizations of Surfaces

We call the raqnge of r the **surface** S defined or traced by r, and 1 together with the domain R constitute a **parametrization** of the surface.

The variables u and v are the **parameters**, and R is defined by inequalities of the form $a \leq u \leq b, c \leq v \leq d$.

The requirement that r be one-to-one on the interior of R ensures that S does not cross itself.

Notice that is the vector equivalent of three parametric equations :

$$x = f(u, v), \quad y = g(u, v), \quad z = h(u, v).$$

Surface Area

Our goal is to find a double integral for calculating the area of a curved surface S based on the parametrization

$$\mathrm{r}(u,v) = f(u,v)\mathrm{i} + g(u,v)\mathrm{j} + h(u,v)\mathrm{k}, \quad a \le u \le b, \quad c \le v \le d,$$

We need to assume that S is smooth enough for the construction we are about to carry out.

The definiton of smoothness involves the partial derivatives of r with respect to u and v:

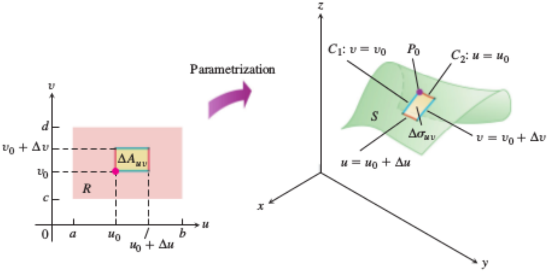
$$\mathrm{r}_u = \frac{\partial \mathrm{r}}{\partial u} = \frac{\partial f}{\partial u}\mathrm{i} + \frac{\partial g}{\partial u}\mathrm{j} + \frac{\partial h}{\partial u}\mathrm{k} \qquad \mathrm{r}_v = \frac{\partial \mathrm{r}}{\partial v} = \frac{\partial f}{\partial v}\mathrm{i} + \frac{\partial g}{\partial v}\mathrm{j} + \frac{\partial h}{\partial v}\mathrm{k}.$$

A parametrized surface

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$$

is **smooth** if $\mathbf{r}_u$ and $\mathbf{r}_v$ are continous and $\mathbf{r}_u \times \mathbf{r}_v$ is never zero on the parameter doamin.

Now consider a small rectangle ΔA_{uv} in R with sides on the lines $u = u_0, u = u_0 + \Delta u, v = v_0$ and $v = v_0 + \Delta v$.

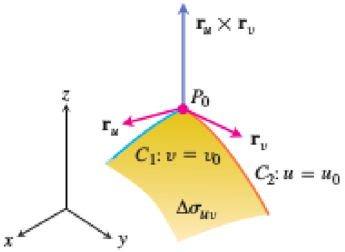


Surface Area

Each side of ΔA_{uv} maps to a curve on the surface S, and together these four curves bound a "curved area element" $\Delta\sigma_{uv}$.

In the notatoin of the figure, the side $v = v_0$ maps to curve C_1, the side $u = u_0$ maps to C_2, and their common vertex (u_0, v_0) maps to P_0.

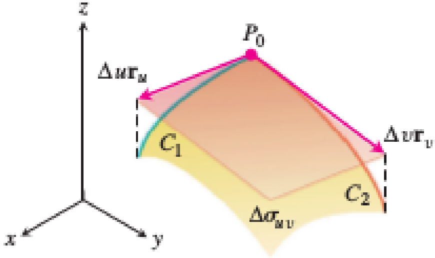
Figure shows an enlarges view of $\Delta\sigma_{uv}$. The vector $\mathrm{r}_u(u_0, v_0)$ is tangent to C_1 at P_0.



Surface Area

Likewise, $\mathrm{r}_v(u_0, v_0)$ is tangent to C_2 at P_0. Here is where we begin to use the assumption that S is smooth. We want to be sure that $\mathrm{r}_v \times \mathrm{r}_v \neq 0$.

We next approximate the surface element $\Delta\sigma_{uv}$ by the parallelogram on the tangent plane whose dises are determined by the vectors $\Delta u \mathrm{r}_u$ and $\Delta v \mathrm{r}_v$.



Surface Area

The area of this parallelogram is

$$|\Delta u \mathbf{r}_u \times \Delta v \mathbf{r}_v| = |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v. \tag{2}$$

A partition of the region R in the uv-plane by rectangular regions ΔA_{uv} generates a partition of the surface S into surface area elements $\Delta \sigma_{uv}$.

Surface Area

We approximate the area of each surface element $\Delta\sigma_{uv}$ by the parallelogram in 2 and sum these areas together to obtain an approximation of the area of S:

$$\sum_u \sum_v |\mathrm{r}_u \times \mathrm{r}_v| \Delta u \Delta v. \tag{3}$$

As Δu and Δv approach zero independently, the continuity of r_u and r_v guarantees that the sum in 3 approaches the double integral

$$\int_c^d \int_a^b |\mathrm{r}_u \times \mathrm{r}_v| \; du \; dv.$$

This double integral gives the area of the surface S.

Parametric Formula for the Area of a Smooth Surface

The area of the smooth surface

$$\mathrm{r}(u,v) = f(u,v)\mathrm{i} + g(u,v)\mathrm{j} + h(u,v)\mathrm{k}, \quad a \le u \le b, \quad c \le v \le D$$

is

$$\int_c^d \int_a^b |\mathrm{r}_u \times \mathrm{r}_v| \; du \; dv.$$

Surface Area Differential :

$$d\sigma = |\mathrm{r}_u \times \mathrm{r}_v| \; du \; dv$$

Differential Formula for Surface Area

$$\iint_S d\sigma.$$

Having found the formula for calculating the area of a parameterized surface, we can now integrate a function over the surface using the parametrized form.

If S is a smooth surface defined parametrically as

$$\mathrm{r}(u,v) = f(u,v)\mathrm{i} + g(u,v)\mathrm{j} + h(u,v)\mathrm{k}, \quad a \leq u \leq b, \quad c \leq v \leq d$$

and $G(x,y,z)$ is a continuous function defined on S, then the **integral of** G **over** S is

$$\iint_S G(x,y,z)\ d\sigma = \int_c^d \int_a^b G\big(f(u,v), g(u,v), h(u,v)\big)\ |\mathrm{r}_u \times \mathrm{r}_v|\ du\ dv.$$

Exercise 1.

In the following exercises, find a parameterization of the surface. (There are many correct ways to do these, so your answers may not be the same as those of others.)

1. *The paraboloid* $z = x^2 + y^2, z \leq 4$
2. Cone frustum : *The first-octant portion of the cone* $z = \sqrt{x^2 + y^2}/2$ *between the planes* $z = 0$ *and* $z = 3$.
3. Cone frustum : *The portion of the cone* $z = 2\sqrt{x^2 + y^2}$ *between the planes* $z = 2$ *and* $z = 4$
4. Spherical cap : *The portion of the sphere* $x^2 + y^2 + z^2 = 4$ *in the first octant between the xy-plane and the cone* $z = \sqrt{x^2 + y^2}$
5. Spherical band : *The portion of the sphere* $x^2 + y^2 + z^2 = 3$ *between the planes* $z = -\sqrt{3}/2$ *and* $z = \sqrt{3}/2$

Solution for Exercise 1

1. In cylindrical coordinates, let $x = r\ \cos\theta, y = r\ sin\theta, z = (\sqrt{x^2+y^2}) = r^2$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r^2\mathbf{k}, 0 \le r \le 2, 0 \le \theta \le 2\pi$.
2. In cylindrical coordinates, let $x = r\cos theta, y = r\sin\theta, z = \frac{\sqrt{x^2+y^2}}{2} \Rightarrow z = \frac{r}{2}$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \left(\frac{r}{2}\right)\mathbf{k}$. For $0 \le z \le 3, 0 \le \frac{r}{2} \le 3 \Rightarrow 0 \le r \le 6$; to get only the first octant, let $0 \le \theta \le \frac{\pi}{2}$.
3. In cylindrical coordintes, let $x = r\cos\theta, y = r\sin\theta; z = 2\sqrt{x^2+y^2} \Rightarrow z = 2r$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + 2r\mathbf{k}$. For $2 \le z \le 4, 2 \le 2r \le 4 \Rightarrow 1 \le \le r \le 2$, and let $0 \le \theta \le 2\pi$.
4. In cylinrical coordinates, $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \sqrt{4-r^2}\mathbf{k}$ (see Exercise 5 above with $x^2+y^2+z^2=4$, instead of $x^2+y^2+z^2=9$). For the first octant, let $0 \le \theta \le \frac{\pi}{2}$. For the domain of $r : z = \sqrt{x^2+y^2}$ and $x^2+y^2+z^2 = 4 \Rightarrow x^2+y^2+(\sqrt{x^2+y^2})^2 = 4 \Rightarrow 2(x^2+y^2) = 4 \Rightarrow 2r^2 = 4 \Rightarrow r = \sqrt{2}$. Thus, let $\sqrt{2} \le r \le 2$(to get the portion of the sphere between the cone and the xy-plane).
5. In spherical coordinates, $x = \rho\sin\phi\cos\theta, y = \rho\sin\theta, \rho = \sqrt{x^2+y^2+z^2} \Rightarrow \rho^2 = 3 \Rightarrow = \sqrt{3} \Rightarrow z = \sqrt{3}\cos\phi$ for the sphere; $z = \frac{\sqrt{3}}{2} = \sqrt{3}\cos\phi \Rightarrow \cos\phi = \frac{1}{2} \Rightarrow \phi = \frac{\pi}{3}; z = -\frac{\sqrt{3}}{2} = \sqrt{3}\cos\phi\frac{\pi}{3} \le \phi \le \frac{2\pi}{3}$ and $0 \le \theta 2\pi$.

Exercise 2.

In the following exercises, find a parameterization of the surface. (There are many correct ways to do these, so your answers may not be the same as those of others.)

1. Parabolic cylinder between planes : *The surface cut from the parabolic cylinder* $z = 4 - y^2$ *by the planes* $x = 0, x = 2$, *and* $z = 0$
2. Circular cylinder band : *The portion of the cylinder* $y^2 + z^2 = 9$ *between the planes* $x = 0$ *and* $x = 3$
3. Titled plane inside cylinder : *The portion of the plane* $x + y + z = 1$
   (a) *Inside the cylinder* $x^2 + y^2 = 9$
   (b) *Inside the cylinder* $y^2 + z^2 = 9$
4. Circular cylinder band : *The portion of the cylinder* $y^2 + (z - 5)^2 = 25$ *between the planes* $x = 0$ *and* $x = 10$

Solution for Exercise 2

1. Since $z = 4 - y^2$, we can let $\mathbf{r}$ be a function of x and $y \Rightarrow \mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (4 - y^2)\mathbf{k}$. Then $z = 0 \Rightarrow 0 = 4 - y^2 \Rightarrow y = \pm 2$. Thus, let $-\sqrt{2} \le x \le \sqrt{2}$ and $0 \le z \le 3$
2. When $x = 0$, let $y^2 + z^2 = 9$ be the circular section in the yz-plane. Use polar coordinates in the yz-plane $\Rightarrow x^2 = 2 \Rightarrow x = \pm 2$. Thus, let $-\sqrt{2} \le x \le \sqrt{2}$ and $0 \le z \le 3$.
3. (a) $x + y + z = 1 \Rightarrow z = 1 - x - y$. In cylindrical coordinates, let $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow z = 1 - r\cos\theta - r\sin\theta \Rightarrow \mathbf{r}(r, \theta) =$ $(r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (1 - r\cos\theta - r\sin\theta)\mathbf{k}, 0 \le \theta \le 2\pi$ and $0 \le r \le 3$.

   (b) In a fashion similar to cylindrical coordinates, but working in the ya-plane instead of the xy-plane, let $y = u\cos v, z = u\sin v$ where $u = \sqrt{y^2 + z^2}$ and v is the angle formed $(x, y, z), (x, 0, 0)$, and $(x, y, 0)$ with $(x, 0, 0)$ as vertex. Since $x + y + z = 1 \Rightarrow x = 1 - y - z \Rightarrow x = 1 - u\cos v - u\sin v$, then $\mathbf{r}$ is a function of u and $v \Rightarrow \mathbf{r}(u, v) = (1 - u\cos v - u\sin v)\mathbf{i} + (u\cos v)\mathbf{j} + (u\sin v)\mathbf{k}, 0 \le u \le 3$ and $0 \le v \le 2\pi$
4. Let $y = w\cos v$ and $z = w\sin v$. Then $y^2 + (z - 5)^2 = 25 \Rightarrow y^2 + z^2 - 10z = 0$ $\Rightarrow w^2\cos^2 v + w^2\sin^2 v - 10w\sin v = 0 \Rightarrow w^2 - 10w\sin v = 0 \Rightarrow w = 0$ or $w = 10\sin v$. Now $w = 0 \Rightarrow y = 0$ and $z = 0$, which is a line not a cylinder. Therefore, let $w = 10\sin v \Rightarrow y = 10\sin v\cos v$ and $z = 10\sin^2 v$. Finally, let $x = u$. Then $\mathbf{r}(u, v) = u\mathbf{i} + (10\sin v\cos v)\mathbf{j} + (10\sin^2 v)\mathbf{k}, 0 \le u \le 10$ and $0 \le v \le \pi$.

Exercise 3.

In the following exercises, use a parametrization to express the area of the surface as a double integral. Then evaluate the integral.

1. Titled plane inside cylinder : *The portion of the plane* $y + 2z = 2$ *inside the cylinder* $x^2 + y^2 = 1$
2. Plane inside cylinder : *The portion of the plane* $z = -x$ *inside the cylinder* $x^2 + y^2 = 4$

Solution for Exercise 3

1. Let $x = r\cos\theta$ and $y = r\sin\theta$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \left(\frac{2-r\sin\theta}{2}\right)\mathbf{k}, 0 \le r \le 1$ and $0 \le \theta \le 2\pi \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} - \left(\frac{\sin\theta}{2}\right)\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} - \left(\frac{r\cos\theta}{2}\right)\mathbf{k}$
$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -\frac{\sin\theta}{2} \\ -r\sin\theta & r\cos\theta & -\frac{r\cos\theta}{2} \end{vmatrix}$
$= \left(\frac{-r\sin\theta\cos\theta}{2} + \frac{(\sin\theta)(r\cos\theta)}{2}\right)\mathbf{i} + \left(\frac{r\sin^2\theta}{2} + \frac{r\cos^2\theta}{2}\right)\mathbf{j} + (r\cos^2\theta + r\sin^2\theta)\mathbf{k} = \frac{r}{2}\mathbf{j} + r\mathbf{k}$
$\Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{\frac{r^2}{4} + r^2} = \frac{\sqrt{5}r}{2} \Rightarrow A = \int_0^{2\pi}\int_0^1 \frac{\sqrt{5}r}{2}\,dr\,d\theta = \int_0^{2\pi}\left[\frac{\sqrt{5}r^2}{4}\right]_0^1 d\theta = \int_0^{2\pi} d\theta = \frac{\pi\sqrt{5}}{2}$
2. Let $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow z = -x = -r\cos\theta, 0 \le r \le 2$ and $0 \le \theta \le 2\pi$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} - (r\cos\theta)k \Rightarrow \mathbf{r}_r$ $= (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} - (\cos\theta)\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} + (r\sin\theta)\mathbf{k}$
$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -\cos\theta \\ -r\sin\theta & r\cos\theta & r\sin\theta \end{vmatrix}$
$= (r\sin^2\theta + r\cos^2\theta)\mathbf{i} + (r\sin\theta\cos\theta - r\sin\theta\cos theta)\mathbf{j} + (r\cos^2\theta + r\sin^2\theta)\mathbf{k} = r\mathbf{i} + r\mathbf{k}$
$\Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{r^2 + r^2} = r\sqrt{2} \Rightarrow A = \int_0^{2\pi}\int_0^2 r\sqrt{2}drd\theta = \int^{2\pi} 0 \left[\frac{r^2\sqrt{2}}{2}\right]_0^2 d\theta =$ $\int_0^{2\pi} 2\sqrt{2}d\theta = 4\pi\sqrt{2}$

Exercise 4.

In the following exercises, use a parametrization to express the area of the surface as a double integral. Then evaluate the integral.

1. Cone frustum : *The portion of the cone* $z = 2\sqrt{x^2 + y^2}$ *between the planes* $z = 2$ *and* $z = 6$
2. Circular cylinder band : *The portion of the cylinder* $x^2 + y^2 = 1$ *between the planes* $z = 1$ *and* $z = 4$
3. Parabolic band : *The portion of the paraboloid* $z = x^2 + y^2$ *between the planes* $z = 1$ *and* $z = 4$

Solution for Exercise 4

1. Let $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow z = 2\sqrt{x^2+y^2} = 2r, 1 \le r \le 3$ and $0 \le \theta \le 2\pi$. Then
$\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + 2r\mathbf{k} \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 2\mathbf{k}$ and
$$\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 2 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-2r\cos\theta)\mathbf{i} - (2r\sin\theta)\mathbf{j} + (r\cos^2\theta + r\sin^2\theta)\mathbf{k}$$
$$= (-2r\cos\theta)\mathbf{i} - (2r\sin\theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + r^2} = \sqrt{5r^2} = r\sqrt{5}$$
$$\Rightarrow A = \int_0^{2\pi} = \int_0^{2\pi} r\sqrt{5}drd\theta = \int_0^{2\pi}\left[\frac{r^2\sqrt{5}}{2}\right]_1^3 d\theta = \int_0^{2\pi} 4\sqrt{5}d\theta = 8\pi\sqrt{5}$$

2. Let $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow r^2 = x^2 + y^2 = 1, 1 \le z \le 4$ and $0 \le \theta \le 2\pi$. Then
$\mathbf{r}(z,\theta) = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + z\mathbf{k} \Rightarrow \mathbf{r}_z = \mathbf{k}$ and $\mathbf{r}_\theta = (-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$
$$\Rightarrow \mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} \Rightarrow |\mathbf{r}_\theta \times \mathbf{r}_z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$
$$\Rightarrow A = \int^{2\pi} 0 \int_1^4 1\ dr\ d\theta = \int_0^{2\pi} 3d\theta = 6\pi$$

3. Let $x = r\cos\theta, y = \sin\theta$ and $z = x^2 + y^2 = r^2$. Then
$\mathbf{r}(r\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r^2\mathbf{k}, 1 \le r \le 2, 0 \le \theta \le 2\pi \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 2r\mathbf{k}$ and
$\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}$
$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-2r^2\cos\theta)\mathbf{i} - (2r^2\sin\theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta|$$
$$= \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2} = r\sqrt{4r^2+1} \Rightarrow A = \int_0^{2\pi}\int_1^2 r\sqrt{4r^2+1}dr\ d\theta = \int_0^{2\pi}\left[\frac{1}{12}(4r^2+1)^{3/2}\right]_1^2 d\theta$$
$$= \int_0^{2\pi}\left(\frac{17\sqrt{17}-5\sqrt{5}}{12}\right)d\theta = \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5})$$

Exercise 5.

In the following exercises, use a parametrization to express the area of the surface as a double integral. Then evaluate the integral.

1. Sawed-off sphere : *The lower portion cut from the sphere* $x^2 + y^2 + z^2 = 2$ *by the cone* $z = \sqrt{x^2 + y^2}$
2. Spherical band : *The portion of the sphere* $x^2 + y^2 + z^2 = 4$ *between the planes* $z = -1$ *and* $z = \sqrt{3}$

1. Let $x = \rho \sin\phi\cos\theta$, $y = \rho\sin\theta$, and $z = \rho\cos\phi \Rightarrow \rho = \sqrt{x^2+y^2+z^2} = \sqrt{2}$ on the sphere. Next, $x^2+y^2+z^2 = 2$ and $z = \sqrt{x^2+y^2} \Rightarrow z^2+z^2 = 2 \Rightarrow z^2 = 1 \Rightarrow z = 1$ since $z \geq 0 \Rightarrow \phi = \frac{\pi}{4}$. For tfhe lowe portion of the sphere cut by the cone, we get $\phi = \pi$. Then
$\mathbf{r}(\phi,\theta) = \left(\sqrt{2}\sin\theta\cos\theta\right)\mathbf{i} + \left(\sqrt{2}\sin\phi\sin\theta\right)\mathbf{j} + \left(\sqrt{2}\cos\phi\right)\mathbf{k},\ \frac{\pi}{4} \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$
$\Rightarrow \mathbf{r}_\theta = \left(\sqrt{2}\cos\theta\cos\theta\right)\mathbf{i} + \left(\sqrt{2}\cos\phi\sin\theta\right)\mathbf{j} - \left(\sqrt{2}\sin\phi\right)\mathbf{k}$, and $\mathbf{r}_\theta = \left(-\sqrt{2}\sin\phi\sin\theta\right)\mathbf{i} + \left(\sqrt{2}\sin\phi\cos\theta\right)\mathbf{j}$
$\Rightarrow \mathbf{r}_\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{2}\cos\phi\cos\theta & \sqrt{2}\cos\phi\sin\theta & -\sqrt{2}\sin\phi \\ -\sqrt{2}\cos\phi\cos\theta & \sqrt{2}\cos\phi\sin\theta & 0 \end{vmatrix}$
$= (2\sin^2\phi\cos\theta)\mathbf{i} + (2\sin^2\phi\sin\theta)\mathbf{j} + (2\sin\phi\cos\phi)\mathbf{k}$
$\Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{4\sin^4\phi\cos^2\theta + 4\sin^4\phi\sin^2\theta + 4\sin^2\phi\cos^2\phi} = \sqrt{4\sin^2\phi} = 2|\sin\phi| = 2\sin\phi$
$\Rightarrow A = \int_0^{2\pi}\int_{\pi/4}^{\pi} 2\sin\phi\, d\phi\, d\theta = \int_0^{2\pi}(2+\sqrt{2})d\theta = (+2+\sqrt{2})\pi$

2. Let $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi \Rightarrow \rho = \sqrt{x^2+y^2+z^2} = 2$ on the sphere. Next, $z = -1 \Rightarrow -1 = 2\cos\phi \Rightarrow \cos\phi = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}$; $z = \sqrt{3} \Rightarrow \sqrt{3} = 2\cos\phi \Rightarrow \cos\phi = \frac{\sqrt{3}2}{\Rightarrow}\phi = \frac{\pi}{6}$. Then
$\mathbf{r}(\phi,\theta) = (2\sin\phi\cos\theta)\mathbf{i} + (2\sin\phi\sin\theta)\mathbf{j} + (2\cos\phi)\mathbf{k},\ \frac{\pi}{6} \leq \phi \leq \frac{2\pi}{3}, 0 \leq \theta \leq 2\pi.$
$\Rightarrow \mathbf{r}_\phi = (2\cos\phi\cos\theta)\mathbf{i} + (2\cos\phi\sin\theta)\mathbf{j} - (2\sin\phi)\mathbf{k}$ and
$\mathbf{r}_\theta = (-2\sin\phi\sin\theta)\mathbf{i} + (2\sin\phi\cos\theta)\mathbf{j}$
$\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix}$
$= (4\sin^2\phi\cos\theta)\mathbf{i} + (4\sin^2\phi\sin\theta)\mathbf{j} + (4\sin\phi\cos\phi)\mathbf{k}$
$\Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{16\sin^4\phi\cos^2\theta + 16\sin^4\phi\sin^2\theta + 16\sin^2\phi\cos^2\phi} = \sqrt{16\sin^2\phi} = 4|\sin\phi| = 4\sin\phi$
$\Rightarrow A = \int_0^{2\pi}\int_{\pi/6}^{2\pi/3} 4\sin\phi\, d\phi\, d\theta = \int_0^{2\pi}(2+2\sqrt{3})d\theta = (4+4\sqrt{3})\pi$

Exercise 6.

The tangent plane at a point $P_0\left(f\left(u_0, v_0\right), g\left(u_0, v_0\right), h\left(u_0, v_0\right)\right)$ *on a parametrized surface* $r(u, v) = f(u, v)\, i + g(u, v)\, j + h(u, v)\, k$ *is the plane through* P_0 *normal to the vector* $r_u\left(u_0, v_0\right) \times r_v\left(u_0, v_0\right)$, *the cross product of the tangent vectors* $r_u\left(u_0, v_0\right)$ *and* $r_v\left(u_0, v_0\right)$ *at* P_0.

In the following exercises, find an equation for the plane tangent to the surface at P_0. *Then find a Cartesian equation for the surface and sketch the surface and tangent plane together.*

1. Cone : *The cone* $r(r, \theta) = (r\cos\theta)\, i + (r\sin\theta)\, j + rk, r \geq 0, 0 \leq \theta \leq 2\pi$ *at the point* $P_0\left(\sqrt{2}, \sqrt{2}, 2\right)$ *corresponding to* $(r, \theta) = (2, \pi/4)$
2. Hemisphere : *The hemisphere surface* $r(\phi, \theta) = (4\sin\phi\cos\theta)\, i + (4\sin\phi\sin\theta)\, j + (4\cos\phi)\, k, 0 \leq \phi \leq \pi/2, 0 \leq \theta \leq 2\pi$, *at the point* $P_0\left(\sqrt{2}, \sqrt{2}, 2\sqrt{3}\right)$ *corresponding to* $(\phi, \theta) = (\pi/6, \pi/4)$

Solution for Exercise 6

1. The parameterization $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$ at
$P_0 = (\sqrt{2}, \sqrt{2}, 2) \Rightarrow \theta = \frac{\pi}{4}, r = 2, \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \mathbf{k} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} + \mathbf{k}$
$\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$
$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{2}/2 & \sqrt{2}/2 & 1 \\ -\sqrt{2} & \sqrt{2} & 0 \end{vmatrix}$
$= -\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j} + 2\mathbf{k} \Rightarrow$ the tangent plane is
$0 = (-\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j} + 2\mathbf{k})\dot{[}(x-\sqrt{2})\mathbf{i} + (y-\sqrt{2})\mathbf{j} + (z-2)\mathbf{k}] \Rightarrow \sqrt{2}x + \sqrt{2}y - 2z = 0$, or $x + y - \sqrt{2}z = 0$.
The parameterization $\mathbf{r}(r,\theta) \Rightarrow x = \cos\theta, y = r\sin\theta$ and $z = r \Rightarrow x^2 + y^2 = r^2 = z^2 \Rightarrow$ the surface is
$z = \sqrt{x^2 + y^2}$.
2. The parameterization $\mathbf{r}(\phi,\theta) = (4\sin\phi\cos\theta)\mathbf{i} + (4\sin\phi\sin\theta)\mathbf{j} + (4\cos\phi)\mathbf{k}$ at
$P_0 = (\sqrt{2}, \sqrt{2}, 2\sqrt{3}) \Rightarrow \rho = 4$ and$z = 2\sqrt{3} = 4\cos\phi \Rightarrow \phi = \frac{\pi}{6}$; and $y = \sqrt{2} \Rightarrow \theta = \frac{\pi}{4}$. Then $\mathbf{r}_\phi$
$= (4\cos\phi\cos\theta)\mathbf{i} + (4\cos\phi\sin\theta)\mathbf{j} - (4\sin\phi)\mathbf{k} = \sqrt{6}\mathbf{i} + \sqrt{6}\mathbf{j} - 2k$ and
$\mathbf{r}_\theta = (-4\sin\phi\sin\theta)\mathbf{i} + (4\sin\phi\cos\theta)\mathbf{j}$
$= -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$ at $P_0 \Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{6} & \sqrt{6} & -2 \\ -\sqrt{2} & \sqrt{2} & 0 \end{vmatrix} = 2\sqrt{2}\mathbf{i} + 2\sqrt{2}j + 4\sqrt{3}\mathbf{k} \Rightarrow$ the tangent plane is
$(2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} + 4\sqrt{3}\mathbf{k}) \cdot \left[(x - \sqrt{2}\mathbf{i} + (y-\sqrt{2})\mathbf{j} + (z - 2\sqrt{3})\mathbf{k}\right] = 0 \Rightarrow \sqrt{2}x + \sqrt{2}y + 2\sqrt{3}z = 16$, or
$x + y + \sqrt{6}z = 8\sqrt{2}$. The parametrization $\Rightarrow x = 4\sin\phi\cos\theta, y = 4\sin\phi\sin\theta, z = 4\cos\phi \Rightarrow$ the surface is
$x^2 + y^2 + z^2 = 16, z \geq 0$.

Exercise 7.

In the following exercises, find an equation for the plane tangent to the surface at P_0. Then find a Cartesian equation for the surface and sketch the surface and tangent plane together.

1. Circular cylinder : *The circular cylinder* $r(\theta, z) = (3\sin 2\theta)\, i + \left(6\sin^2\theta\right) j + zk, 0 \leq \theta \leq \pi$, *at the point* $P_0\left(3\sqrt{3}/2, 9/2, 0\right)$ *corresponding to* $(\theta, z) = (\pi/3, 0)$ *(See Example 3.)*
2. Parabolic cylinder : *The parabolic cylinder surface* $r(x, y) = xi + yj - x^2 k, -\infty < x < \infty, -\infty < y < \infty$, *at the point* $P_0(1, 2, -1)$ *corresponding to* $(x, y) = (1, 2)$

Solution for Exercise 7

1. The parametrization $\mathbf{r}(\theta, z) = (3\sin 2\theta)\mathbf{i} + (6\sin^2\theta)\mathbf{j} + z\mathbf{k}$ at
$P_0 = \left(\frac{3\sqrt{3}}{2}, \frac{9}{2}, 0\right) \Rightarrow = \theta = \frac{\pi}{3}$ and $z = 0$. Then $\mathbf{r}_\theta = (6\cos 2\theta)\mathbf{i} + (12\sin\theta\cos\theta)\mathbf{j}$
$= -3\mathbf{i} + 3\sqrt{3}\mathbf{j}$ aned $\mathbf{r}_z = \mathbf{k}$ at $P_0 = -3\mathbf{i} + 3\sqrt{3}\mathbf{j}$ and $\mathbf{r}_z = \mathbf{k}$ at P_0
$\Rightarrow \mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 3\sqrt{3} & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3\sqrt{3}\mathbf{i} + 3\mathbf{j} \Rightarrow$ the tangent plane is
$\left(\sqrt{3}x + y = 9.\right)$ The parametrization $\Rightarrow x = 3\sin 2\theta$
and $y = 6\sin^2\theta \Rightarrow x^2 + y^2 = 9\sin^2 2\theta + (6\sin^2\theta)^3$
$= 9(4\sin^2\theta\cos^2\theta) + 36\sin^4\theta = 6(6\sin^2\theta) = 6y \Rightarrow x^2 + y^2 - 6y + 9 = 9 \Rightarrow x^2 + (y-3)^2 = 9$
2. The parametrization $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} - x^2\mathbf{k}$ at
$P_0 = (1, 2, -1) \Rightarrow \mathbf{r}_x = \mathbf{i} - 2x\mathbf{k} = \mathbf{i} - 2\mathbf{k}$ and $\mathbf{r}_y = \mathbf{j}$ at P_0
$\Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{vmatrix} = 2\mathbf{i} + \mathbf{k} \Rightarrow$ the tangent plane is
$(2\mathbf{i} + \mathbf{k}) \cdot [(x-1)\mathbf{i} + (y-2)\mathbf{j} + (z+1)\mathbf{k}] = 0$
$\Rightarrow 2x + z = 1.$
The parametrization $\Rightarrow x = x, y = y$ and $z = -x^2 \Rightarrow$ the surface is $z = -x^2$

Exercise 8.

(a) *A torus of revolution (doughnut) is obtained by rotating a circle C in the xz-plane about the z-axis in space. (See the accompanying figure). If C has radius* $r > 0$ *and center* $(R, 0, 0)$, *show that a parametrization of the torus is*

$$r\,(u, v) = ((R + r\cos u)\cos v)\, i + ((R + r\cos u)\sin v)\, j + (r\sin u)\, k,$$

where $0 \leq u \leq 2\pi$ *and* $0 \leq v \leq 2\pi$ *are the angles in the figure.*

(b) *Show that the surface area of the torus is* $A = 4\pi^2 Rr$.

Solution for Exercise 8

(a) An arbitrary point on the circle C is $x, z) = (R + r\cos u, r\sin u) \Rightarrow (x, y, z)$ is on the trus with $x = (R + r\cos u)\cos v, y = (R + r\cos u)\sin v$, adn $z = r\sin u, 0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi$

(b) $\mathbf{r}_u = (-r\sin u\cos v)\mathbf{i} - (r\sin u\sin v)\mathbf{j} + (r\cos u)\mathbf{k}$ and
$\mathbf{r}_v = (-(R + r\cos u)\sin v)\mathbf{i} + ((R + r\cos u)\cos v)\mathbf{j}$

$$\Rightarrow \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin u\cos v & -r\sin u\sin v & r\cos u \\ -(R + r\cos u)\sin v & (R + r\cos u)\cos v & 0 \end{vmatrix}$$

$= -(R + r\cos u)(r\cos v\cos u)\mathbf{i} - (R + r\cos u)(r\sin v\cos u)\mathbf{j} + (-r\sin u)(R + r\cos u)\mathbf{k}$
$\Rightarrow |\mathbf{r}_u \times \mathbf{r}_v|^2 = (R + r\cos u)^2(r^2\cos^2 v\cos^2 u + r^2\sin^2 v\cos^2 u + r^2\sin^2 u) \Rightarrow |\mathbf{r}_u \times \mathbf{r}_v| = r(R + r\cos u)$
$\Rightarrow A = \int_0^{2\pi}\int_0^{2\pi}(rR + r^2\cos u)dudv = \int_0^{2\pi} 2\pi R\ dv = 4\pi^2 rR$

Exercise 9.

Suppose that the parametrized curve $C : (f(u), g(u))$ *is revolved about the x-axis, where* $g(u) > 0$ *for* $a \leq u \leq b$.

(a) *Show that*

$$r(u, v) = f(u)\, i + (g(u) \cos v)\, j + (g(u) \sin v)\, k$$

Is a parameterization of the resulting surface of revolution, where $0 \leq v \leq 2\pi$ *is the angle from the xy-plane to the point* $r(u, v)$ *on the surface. (See the accompanying figure). Notice that* $f(u)$ *measures distance along the axis of revolution and* $g(u)$ *measures distance from the axis of revolution.*

(b) *Find a parametrization for the surface obtained by revolving the curve* $x = y^2, y \geq 0$, *about the x-axis*

Solution for Exercise 9

(a) The point (x, y, z) is on the surface for fixed $x = f(u)$ when $y = g(u)\sin\left(\frac{\pi}{2} - v\right)$ and $z = g(u)\cos\left(\frac{\pi}{2} - v\right) \Rightarrow x = f(u), y = g(u)\cos v$, and $z = g(u)\sin v \Rightarrow \mathbf{r}(u, v) = f(u)\mathbf{i} + (g(u)\cos v)\mathbf{j} + (g(u)\sin v)\mathbf{k}, 0 \leq v \leq 2\pi, a \leq u \leq b$

(b) Let $u = y$ and $x = u^2 \Rightarrow f(u) = u^2$ and $g(u) = u \Rightarrow \mathbf{r}(u, v) = u^2\mathbf{i} + (u\cos v)\mathbf{j} + (u\sin v)\mathbf{k}, 0 \leq v \leq 2\pi, 0 \leq u.$

Exercise 10.

(a) *Recall the parametrization* $x = a\cos\theta, y = b\sin\theta, 0 \leq \theta \leq 2\pi$ *for the ellipse* $\left(x^2/a^2\right) + \left(y^2/b^2\right) =$*(Section 3.9, Example 5). Using the angles* θ *and* ϕ *in spherical coordinates, show that* $r(\theta, \phi) = (a\cos\theta\cos\phi)\, i + (b\sin\theta\cos\phi)\, j + (c\sin\phi)\, k$ *is a parametrization of the ellipsoid* $\left(x^2/a^2\right) + \left(y^2/b^2\right) + \left(z^2/c^2\right) = 1.$

(b) *Write an integral for the surface are of the ellipsoid, but do not evaluate the integral.*

Solution for Exercise 10

(a) Let $w^2 \frac{z^2}{c^2} = 1$ where $w = \cos\phi$ and $\frac{z}{c} = \sin\phi \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\phi \Rightarrow \frac{x}{a} = \cos\phi\cos\theta$ and $\frac{y}{b} = \cos\phi\sin\theta$
$\Rightarrow x = a\ \cos\theta\cos\phi, y = b\sin\theta\cos\phi,$ and $z = c\sin\phi$
$\Rightarrow \mathbf{r}(\theta,\phi) = (a\ \cos\theta\cos\phi)\mathbf{i} + (b\sin\theta\cos\theta)\mathbf{j} + (c\sin\phi)\mathbf{k}$

(b) $\mathbf{r}_\theta = (-a\sin\theta\cos\phi)\mathbf{i} + (b\cos\theta\cos\phi)\mathbf{j}$ and
$r_\phi = (-a\cos\theta\sin\phi)\mathbf{i} - (b\sin\theta\sin\phi)\mathbf{j} + (c\cos\phi)\mathbf{k}$

$$\Rightarrow \mathbf{r}_\theta \times \mathbf{r}_\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin\theta\cos\phi & b\cos\theta\cos\phi & 0 \\ -a\cos\theta\sin\phi & -b\sin\theta\sin\phi & c\cos\phi \end{vmatrix}$$

$= (bc\cos\theta\cos^2\phi)\mathbf{i} + (ac\sin\theta\cos^2\phi)\mathbf{j} + (ab\sin\phi\cos\phi)\mathbf{k}$
$\Rightarrow |\mathbf{r}_\theta \times \mathbf{r}_\phi|^2 = b^2c^2\cos^2\theta\cos^4\phi + a^2c^2\sin^2\theta\cos^4\phi + a^2b^2\sin^2\phi\cos^2\phi$, and the result follows. $A \Rightarrow \int_0^{2\pi}\int_0^{2\pi} |\mathbf{r}_\theta \times \mathbf{r}_\phi| d\phi d\theta =$
$\int_0^{2\pi}\int_0^{\pi} [a^2b^2\sin^2\phi\cos^2\phi + b^2c^2\cos^2\theta\cos^4\phi + a^2c^2\sin^2\theta\cos^4\phi]^{[1/2]} d\phi\ d\theta$

Hyperboloid of one sheet

Exercise 11.

(a) *Find a parametrization for the hyperboloid of one sheet* $x^2 + y^2 - z^2 = 1$ *in terms of the angle* θ *associated with the circle* $x^2 + y^2 = r^2$ *and the hyperbolic parameter* u *associated with the hyperbolic function* $r^2 - z^2 = 1$.*(Hint:* $\cosh^2 u - \sin h^2 u = 1$.*)*

(b) *Generalize the result in part (a) to the hyperboloid* $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$.

Solution for Exercise 11

(a) $\mathbf{r}(\theta, u) = (\cosh u \cos\theta)\mathbf{i} + (\cosh u \sin\theta)\mathbf{j} + (\sinh \quad u)\mathbf{k}$

(b) $\mathbf{r}(\theta, u) = (a \cosh u \cos\theta)\mathbf{i} + (5 \cosh u \sin\theta)\mathbf{j} + (c \sinh u)\mathbf{k}$

Exercise 12.

1. *(Continuation of the above exercise.) Find a Cartesian equation for the plane tangent to the hyperboloid* $x^2 + y^2 - z^2 = 25$ *as the point* $(x_0, y_0, 0)$ *, where* $x_0^2 + y_0^2 = 25$.
2. Hyperboloid of two sheets : *Find a parametrization of the hyperboloid of two sheets* $\left(z^2/c^2\right) - \left(x^2/a^2\right) - \left(y^2/b^2\right) = 1$.

Solution for Exercise 12

1. $\mathbf{r}(\theta, u) = (5\cosh u \cos\theta)\mathbf{i} + (5\cosh u \sin\theta)\mathbf{j} + (5\sinh u)\mathbf{k} \Rightarrow \mathbf{r}_\theta =$ $(-5\cosh u \sin\theta)\mathbf{i} + (5\cosh u \cos\theta)\mathbf{j}$ and $\mathbf{r}_u = (5\sinh u \cos\theta)\mathbf{i} + (5\sinh u \sin\theta)\mathbf{j} + (5\cosh u)\mathbf{k}$
$$\Rightarrow \mathbf{r}_\theta \times \mathbf{r}_u = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5\cosh u \sin\theta & 5\cosh u \cos\theta & 0 \\ 5\sinh u \cos\theta & 5\sinh u \sin\theta & 5\cosh u \end{vmatrix}$$
$= (25\cosh^2 u \cos\theta)\mathbf{i} + (25\cosh^2 u \sin\theta)\mathbf{j} - (25\cosh u \sinh u)\mathbf{k}$. At the point $(x_0, y_0, 0)$, where $x_0^2 + y_0^2 = 25$ we have $5\sinh u = 0 \Rightarrow u = 0$ and $x_0 = 25\cos\theta, y_0 = 25\sin\theta \Rightarrow$ the tangent plane is
$5(x_0\mathbf{i} + y_0\mathbf{j}) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + z\mathbf{k}] = 0 \Rightarrow x_0 x - x_0^2 + y_0 y - y_0^2 = 0 \Rightarrow x_0 x + y_0 y = 25$
2. Let $\frac{z^2}{c^2} - w^2 = 1$ where $\frac{z}{c} = \cosh u$ and $w = \sinh u \Rightarrow w^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} \Rightarrow \frac{x}{a} = w\cos\theta$ and $\frac{y}{b} = w\sin\theta$
$\Rightarrow x = a\sinh u \cos\theta, y = b\sinh u \sin\theta,$ and $z = c\cosh u$
$\Rightarrow \mathbf{r}(\theta, u) = (a\sinh u \cos\theta)\mathbf{i} + (b\sinh u \sin\theta)\mathbf{j} + (c\cosh u)\mathbf{k}, 0 \le \theta \le 2\pi, -\infty < u < \infty$

Exercise 13.

1. *Find the area of the surface cut from the paraboloid* $x^2 + y^2 - z = 0$ *by the plane* $z = 2$*.*
2. *Find the area of the portion of the surface* $x^2 - 2z = 0$ *that lies above the triangle bounded by the lines* $x = \sqrt{3}, y = 0$*, and* $y = 3$ *in the xy-plane.*
3. *Find the area of the cap cut from the sphere* $x^2 + y^2 + z^2 = 2$ *by the cone* $z = \sqrt{x^2 + y^2}$*.*

Solution for Exercise 13

1. $\mathbf{p} = \mathbf{k}, \nabla f = 2x\mathbf{i} + 2y\mathbf{j}\text{-}\mathbf{k} \Rightarrow |\nabla f| = \sqrt{(2x)^2 + (2y)^2 + (-1)^2} = \sqrt{4x^2 + 4y^2 + 1}$ and $|\nabla f \cdot \mathbf{f}| = 1; z = 2 \Rightarrow x^2 + y^2 = 2;$thus $S = \int_R \int \frac{|\nabla f|}{|\nabla f \cdot p|} dA = \int_R \int \sqrt{4x^2 + 4y^2 + 1} dx\ dy$
$= \int_R \int \sqrt{4r^2 \cos\theta + 4r^2 \sin^2\theta + 1} r dr\ d\theta = \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{4r^2 + 1} r\ dr\ d\theta =$
$\int_0^{2\pi} \left[\frac{1}{12}(4r^2 + 1)^{3/2}\right] d\theta$
$= \int_0^{2\pi} \frac{13}{6} d\theta = \frac{13}{3}\pi$
2. $\mathbf{p} = \mathbf{k}, \nabla f = 2x\mathbf{i} - 2\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4} = 2\sqrt{x^2 + 1}$ and $|\nabla f \cdot \mathbf{p}| = 2 \Rightarrow S = \int_R \int \frac{|}{\nabla f|} |\nabla f . \mathbf{p}| dA$
$= \int_R \int \frac{2\sqrt{x^2+1}}{2} dxdy = \int_0^{\sqrt{3}} \int_0^x \sqrt{x^2 + 1} dydx = \int_0^{\sqrt{3}} x\sqrt{x^2 + 1} dx = \left[\frac{1}{3}(x^2 + 1)^{3/2}\right]_0^{\sqrt{3}} =$
$\frac{1}{3}(4)^{3/2} - \frac{1}{3} = \frac{7}{3}$
3. $\mathbf{p} = \mathbf{k}, \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = \sqrt{8} = 2\sqrt{2}$ and $|\nabla f . \mathbf{p}| = 2z; x^2 + y^2 + z^2 = 2$ and $z = \sqrt{x^2 + y^2} \Rightarrow x^2 + y^2 = 1;$ thus,
$S = \int_R \int \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \int_R \int \frac{2\sqrt{2}}{2z} dA = \sqrt{2} \int_R \int \frac{1}{z} dA$
$= \sqrt{2} \int_R \int \frac{1}{\sqrt{2-(x^2+y^2)}} dA = \sqrt{2} \int_0^{2\pi} \int_0^1 \frac{r\ dr\ d\theta}{\sqrt{2-r^2}} = \sqrt{2} \int_0^{2\pi} (-1 + \sqrt{2}) d\theta = 2\pi(2 - \sqrt{2})$

Exercise 14.

1. *Find the area of the portion of the paraboloid* $x = 4 - y^2 - z^2$ *that lies above the ring* $1 \leq y^2 + z^2 \leq 4$ *in the yz-plane.*
2. *Find the area of the surface* $x^2 - 2\ln x + \sqrt{15}y - z = 0$ *above the square* $R : 1 \leq x \leq 2, 0 \leq y \leq 1$, *in the xy-plane.*
3. *Find the area of the surface* $2x^{3/2} + 2y^{3/2} - 3z = 0$ *above the square* $R : 0 \leq x \leq 1, 0 \leq y \leq 1$, *in the sy-plane.*

Solution for Exercise 14

1. $\mathbf{p} = \mathbf{i}, \nabla f = \mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{1^2 + (2y)^2 + (2a)^2} = \sqrt{1 + 4y^2 + 4z^2}$ and $|\nabla f \cdot \mathbf{p}| = 1 : 1 \leq y^2 + z^2 \leq 4 \Rightarrow S = \int_R \int \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \int_R \int \sqrt{1 + 4y^2 + 4z^2} dy\ dz =$
$\int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2 \cos^2\theta + 4r^2 \sin^2\theta} r\ dr\ d\theta$
$= \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2} r\ dr\ d\theta = \int_0^{2\pi} \left[\frac{1}{12}(1 + 4r^2)^{3/2}\right]_1^1 d\theta = \int_0^{2\pi} \frac{1}{12}(17\sqrt{17} - 5\sqrt{5}) d\theta =$
$\frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5})$

2. $\mathbf{p} = \mathbf{k}, \nabla f = (2x - \frac{2}{x})\mathbf{i} + \sqrt{15}\mathbf{j}\text{-}\mathbf{k} \Rightarrow |\nabla f| = \sqrt{\left(2x - \frac{2}{x}\right)^2 + (\sqrt{15})^2 + (-1)^2} =$
$\sqrt{4x^2 + 8 + \frac{4}{x^2}} = \sqrt{(2x + \frac{2}{x})^2} = 2x + \frac{2}{x}$, on $1 \leq x \leq 2$ and $|\nabla f.\mathbf{p}| = 1$
$\Rightarrow S = \int_R \int \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \int_R \int (2x + 2x^{-1}) dx\ dy$
$= \int_0^1 \int_1^2 (2x + 2x^{-1} dx\ dy = \int_0^1 [x^2 + 2\ \ln x]_1^2 dy = \int_0^1 (3 + 2\ \ln 2) dy = 3 + 2\ \ln\ 2$

3. $\mathbf{p} = \mathbf{k}, \nabla f = 3\sqrt{x}\mathbf{i} + 3\sqrt{y}\mathbf{j} - 3\mathbf{k} \Rightarrow |\nabla f| = \sqrt{9x + 9y + 9} = 3\sqrt{x + y + 1}$ and $|\nabla f.\mathbf{p}| = 3$
$\Rightarrow S = \int_R \int \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \int_R \int \sqrt{x + y + 1} dx\ dy = \int_0^1 \int_0^1 \sqrt{x + y + 1} dx\ dy =$
$\int_0^1 \left[\frac{2}{3}(x + y + 1)^{3/2}\right]_0^1 dy$
$= \int_0^1 \left[\frac{2}{3}(y + 2)^{3/2} - \frac{2}{3}(y + 1)^{3/2}\right] dy = \left[\frac{4}{15}(y + 2)^{5/2} - \frac{4}{15}(y + 1)^{5/2}\right]_0^1 =$
$\frac{4}{15}\left[(3)^{5/2} - (2)^{5/2} - (2)^{5/2} + 1\right] = \frac{4}{15}(9\sqrt{3} - 8\sqrt{2} + 1)$

Exercise 15.

Find the area of the surfaces in the following exercises.

1. *The surface cut from the bottom of the paraboloid* $z = x^2 + y^2$ *by the plane* $z = 3$
2. *The portion of the cone* $z = \sqrt{x^2 + y^2}$ *that lies over the region between the circle* $x^2 + y^2 = 1$ *and the ellipse* $9x^2 + 4y^2 = 36$ *in the* xy*-plane . (Hint: Use formulas from geometry to find the area of the region.)*
3. *The Surface in the first octant from the cylinder* $y = (2/3)\, z^{3/2}$ *by the planes* $x = 1$ *and* $y = 16/3$.
4. *The portion of the plane* $y + z = 4$ *that lies above the region cut from the first quadrant of the* xz*-plane by the parabola* $x = 4 - z^2$

Solution for Exercise 15

1. $f_x(s, y) = 2x, f_y(x, y) = 2y \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{4x^2 + 4y^2 + 1} \Rightarrow$ Area $=$ $\int_R \int \sqrt{4x^2 + 4y^2 + 1} dx\ dy =$ $\int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{4r^2 + 1} r\ dr\ d\theta = \frac{\pi}{6}(13\sqrt{13} - 1)$
2. $f_x(x, y) = \frac{x}{\sqrt{x^2+y^2}}, f_y(X, y) = \frac{y}{\sqrt{x^2+y^2}} \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1} =$ $\sqrt{2} \Rightarrow$ Area $\int_{R_{xy}} \int \sqrt{2} dx\ dy = \sqrt{2}$(area between the ellipse and the circle)$= \sqrt{2}(6\pi - \pi) = 5\pi\sqrt{2}$
3. $y = \frac{2}{3} z^{3/2} \Rightarrow f_x(x, z) = 0, f_z(x, a) = z^{1/2} \Rightarrow \sqrt{f_x^2 + f_z^2 + 1} = \sqrt{z + 1}; y = \frac{16}{3} \Rightarrow \frac{13}{3} =$ $\frac{2}{3} z^{3/2} \Rightarrow z = 4 \Rightarrow$ Area $= \int_0^4 \int_0^1 \sqrt{z + 1} dx\ dz = \int_0^4 \sqrt{z + 1} sz = \frac{2}{3}(5\sqrt{5} - 1)$
4. $y = 4 - z \Rightarrow f_x(x, z) = 0, f_z(x, a) = -1 \Rightarrow \sqrt{f_x^2 + f_z^2 + 1} = \sqrt{2} \Rightarrow$ Ara $=$ $\int_{R_{xz}} \int \sqrt{2} dA = \int_0^2 \int_0^{4-z^2} \sqrt{2} dx\ dz = \sqrt{2} \int_0^2 (4 - z^2) dz = \frac{16\sqrt{2}}{3}$

Exercise 16.

1. *Use the parametrization*

$$r(x,z) = xi + f(x,z)\,j + zk$$

And Equation (5) *to device a formula for* $d\sigma$ *associated with the explicit from* $y = f(x,z)$.

2. *Let* S *be the surface obtain by rotating the smooth curve* $y = f(x)$, $a \le x \le b$, *about the* x*-axis, where* $f(x) \ge 0$.

   (a) *Show that the vector function*

$$r(x,\theta) = xi + f(x)\cos\theta j + f(x)\sin\theta k$$

   Is a parametrization of S, *where* θ *is tha angle of rotation around the* x*-axis (see accompanying figure).*

   (b) *Use Equation* (4) *to show that the surface area of this surface of revolution is given by*

$$A = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx.$$

Solution for Exercise 16

1. $\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + f(x,y)\mathbf{k} \Rightarrow \mathbf{r}_x(x,y) = \mathbf{i} + f_x(x,y)\mathbf{k}, \mathbf{r}_y(x,y) = \mathbf{j} + f_y(x,y)\mathbf{k}$

   $\Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{jk} & \\ 1 & 0 & f_x(x,y) \\ 0 & 1 & f_y(x,y) \end{vmatrix} = -f_x(x,y)\mathbf{i} - f_y(x,y)\mathbf{j} + \mathbf{k}$

   $\Rightarrow |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{(-f_x(x,y))^2 + (-f_y(x,y))^2 + 1^2} = \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}$

   $\Rightarrow d\sigma = \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1dA}$

2. S is obtained by rotating $y = f(x)$, $a \leq x \leq b$ about the x-axis where $f(x) \geq 0$

   (a) Let (x,y,z) be a point on S. Consider the cross section when $x = x^\star$, the cross section is a circle with radius $r = f(x^\star)$. The set of parametric equation for this circle are given by $y(\theta) = r\cos\theta = f(x^\star)\cos\theta$ and $z(\theta = r\sin\theta = f(x^\star)\sin\theta$ where $0 \leq \theta \leq 2\pi$. Since x can take on any value between a and b we have $x(x,\theta) = x, y(x,\theta) = f(x)\cos\theta, z(x,\theta) = f(x)\sin\theta$ where $a \leq x\ leb$ and $0 \leq \theta \leq 2\pi$. Thus $\mathbf{r}(x,\theta) = x\mathbf{i} + f(x)\cos\theta\mathbf{j} + f(x)\sin\theta\mathbf{k} = f(x)\cos\theta, z(x,\theta) = f(x)\sin\theta$ where $a \leq x \leq b$ and $0 \leq \theta \leq 2\pi$. Thus $\mathbf{r}(x,\theta) = x^*\mathbf{i} + f(x)\cos\theta\mathbf{j} + f(x)\sin\theta\mathbf{k}$

   (b) $\mathbf{r}(x,\theta) = \mathbf{i} + f'(x)\cos\theta\mathbf{j} + f'(x)\sin\theta\mathbf{k}$ and $\mathbf{r}_\theta(x,\theta) = -f(x)\sin\theta\mathbf{j} + f(x)\cos\theta\mathbf{k}$

   $\Rightarrow \mathbf{r}_x \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(x)\cos\theta & f'(x)\sin\theta \\ 0 & -f(x)\sin\theta & f(x)\cos\theta \end{vmatrix} = f(x)\cdot f'(x)\mathbf{i} - f(x)\cos\theta\mathbf{j} - f(x)\sin\theta\mathbf{k}$

   $\Rightarrow |\mathbf{r}_x \times \mathbf{r}_\theta| = \sqrt{(f(x)\cdot f'(x))^2 + (-f(X)\cos\theta)^2 + (-f(x)\sin\theta)^2} = f(x)\sqrt{1 + (f'(x))^2}$

   $A = \int_a^b \int_0^{2\pi} f(x)\sqrt{1 + (f'(x))^2}d\theta dx = \int_a^b \left[\left(f(x)\sqrt{1 + (f'(x))^2}\right)\theta\right]_0^{2\pi} dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}dx$

Exercise 17.

In the following exercises, integrate the given function over the given surface.

1. Parabolic cylinder : $G(x, y, z) = x$, *over the parabolic cylinder* $y = x^2$, $0 \leq x \leq 2$, $0 \leq z \leq 3$
2. Sphere : $G(x, y, z) = x^2$, *over the unit sphere* $x^2 + y^2 + z^2 = 1$

Solution for Exercise 17

1. Let the parametrization be $\mathbf{r}(x,z) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k} \Rightarrow \mathbf{r}_x = \mathbf{i} + 2x\mathbf{j}$ and $\mathbf{r}_z = \mathbf{k} \Rightarrow \mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2x\mathbf{i} + \mathbf{j} \Rightarrow |\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{4x^2+1} \Rightarrow \iint\limits_S G(x,y,z)d\sigma =$
$\int_0^3 \int_0^2 x\sqrt{4x^2+1}dxdz = \int_0^3 \left[\frac{1}{12}(4x^2+1)^{3/2}\right]_0^2 dz = \int_0^3 \frac{1}{12}\left(17\sqrt{17}-1\right) dz = \frac{17\sqrt{17}-1}{4}$
2. Let the parametrization be $\mathbf{r}(\phi,\theta) = (\sin\phi\cos\theta)\mathbf{i} + (\sin\phi\sin\theta)\mathbf{j} + (\cos\phi)\mathbf{k}$ (spherical coordinates with $\rho = 1$ on the sphere),
$0 \le \phi \le \pi, 0 \le \theta \le 2\pi \Rightarrow \mathbf{r}_\phi = (\cos\phi\cos\theta)\mathbf{i} + (\cos\phi\sin\theta)\mathbf{j} - (\sin\phi)\mathbf{k}$ and
$\mathbf{r}_\theta = (-\sin\phi\sin\theta)\mathbf{i} + (\sin\phi\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix}$
$= (\sin^2\phi\cos\theta)\mathbf{i} + (\sin^2\phi\sin\theta)\mathbf{j} + (\sin\phi\cos\phi)\mathbf{k} \Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| =$
$\sqrt{\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \sin^2\phi\cos^2\phi} = \sin\phi$;
$x = \sin\phi\cos\theta \Rightarrow G(x,y,z) = \cos^2\theta\sin^2\phi \Rightarrow \iint\limits_S G(x,y,z)d\sigma =$
$\int_0^{2\pi}\int_0^{\pi}(\cos^2\theta\sin^2\phi)(\sin\phi)d\phi d\theta = \int_0^{2\pi}\int_0^{\pi}(cos^2\theta)(1-\cos^2\phi)(\sin\phi)d\phi d\theta$;
$\begin{bmatrix} u = \cos\phi \\ du = -\sin\phi d\phi \end{bmatrix} \to \int_0^{2\pi}\int_1^{-1}(\cos^2\theta)(u^2-1)dud\theta = \int_0^{2\pi}(\cos^2\theta)\left[\frac{u^3}{3} - u\right]_1^{-1} d\theta =$
$\frac{4}{3}\int_0^{2\pi}\cos^2\theta d\theta = \frac{4}{3}\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi} = \frac{4\pi}{3}$

Exercise 18.

In the following exercises, integrate the given function over the given surface.

1. Portion of plane : $F(x, y, z) = z$, *over the portion of the plane* $x + y + z = 4$ *that lies above the square* $0 \leq x \leq 1$, $0 \leq y \leq 1$, *in the xy-plane.*
2. Cone : $F(x, y, z) = z - x$, *over the cone* $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq 1$.

Solution for Exercise 18

1. Let the parametrization be $\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + (4-x-y)\mathbf{k} \Rightarrow \mathbf{r}_x = \mathbf{i} - \mathbf{k}$ and $\mathbf{r}_y = \mathbf{j} - \mathbf{k}$
$\Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{3} \Rightarrow \iint\limits_S F(x,y,z)d\sigma =$
$\int_0^1 \int_0^1 (4-x-y)\sqrt{3}dydx$
$= \int_0^1 \sqrt{3}\left[4y - xy - \frac{y^2}{2}\right]_0^1 dx = \int_0^1 \sqrt{3}\left(\frac{7}{2} - x\right)dx = \sqrt{3}\left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^1 = 3\sqrt{3}$
2. Let the parametrization be $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $0 \leq r \leq 1$ (since $0 \leq z \leq 1$) and $0 \leq \theta \leq 2\pi$
$\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \mathbf{k}$ and
$\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$
$= (-r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{(-r\cos\theta)^2 + (-r\sin\theta)^2 + r^2} = r\sqrt{2}$; $z = r$ and $x = r\cos\theta$
$\Rightarrow F(x,y,z) = r - r\cos\theta \Rightarrow \iint\limits_S F(x,y,z)d\sigma = \int_0^{2\pi}\int_0^1 (r - r\cos\theta)\left(r\sqrt{2}\right)drd\theta =$
$\sqrt{2}\int_0^{2\pi}\int_0^1 (1-\cos\theta)r^2 drd\theta = \frac{2\pi\sqrt{2}}{3}$

Spherical cap

Exercise 19.

Integrate $H(x, y, z) = yz$*, over the part of the sphere* $x^2 + y^2 + z^2 = 4$ *that lies above the cone* $z = \sqrt{x^2 + y^2}$*.*

Solution for Exercise 19

Let the parametrization be $\mathbf{r}(\phi,\theta) = (2\sin\phi\cos\theta)\mathbf{i} + (2\sin\phi\sin\theta)\mathbf{j} + (2\cos\phi)\mathbf{k}$ (spherical coordinates with $\rho = 2$ on the sphere), $0 \le \phi \le \frac{\pi}{4}$; $x^2 + y^2 + z^2 = 4$ and
$z = \sqrt{x^2+y^2} \Rightarrow z^2 + z^2 = 4 \Rightarrow z^2 = 2 \Rightarrow z = \sqrt{2}$ (since $z \ge 0$)
$\Rightarrow 2\cos\phi = \sqrt{2} \Rightarrow \cos\phi = \frac{\sqrt{2}}{2} \Rightarrow \phi = \frac{\pi}{4}$, $0 \le \theta \le 2\pi$;
$\mathbf{r}_\phi = (2\cos\phi\cos\theta)\mathbf{i} + (2\cos\phi\sin\theta)\mathbf{j} - (2\sin\phi)\mathbf{k}$ and

$$r_\theta = (-2\sin\phi\sin\theta)\mathbf{i} + (2\sin\phi\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix}$$

$= (4\sin^2\phi\cos\theta)\mathbf{i} + (4\sin^2\phi\sin\theta)\mathbf{j} + (4\sin\phi\cos\phi)\mathbf{k}$
$\Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{16\sin^4\phi\cos^2\theta + 16\sin^4\phi\sin^2\theta + 16\sin^2\phi\cos^2\phi} = 4\sin\phi$; $y = 2\sin\phi\sin\theta$ and

$z = 2\cos\phi \Rightarrow H(x,y,z) = 4\cos\phi\sin\phi\sin\theta \Rightarrow \iint_S H(x,y,z)\,d\sigma =$
$\int_0^{2\pi}\int_0^{\pi/4}(4\cos\phi\sin\phi\sin\theta)(4\sin\phi)\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/4} 16\sin^2\phi\cos\phi\sin\theta\,d\phi\,d\theta = 0.$

Exercise 20.

In the following exercises, use a parametrization to find the flux

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

across the surface in the given direction.

1. Parabolic cylinder : $\mathbf{F} = z^2\mathbf{i} + x\mathbf{j} - 3z\mathbf{k}$ *outward (normal away from the x-axis) through the surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x = 0$, $x = 1$, and $z = 0$*
2. Parabolic cylinder : $\mathbf{F} = x^2\mathbf{j} - xz\mathbf{k}$ *outward (normal away from the yz-plane) through the surface cut from the parabolic cylinder $y = x^2$, $-1 \le x \le 1$, by the planes $z = 0$ and $z = 2$*

Solution for Exercise 20

1. Let the parametrization be $\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + (4-y^2)\mathbf{k}$, $0 \le x \le 1$, $-2 \le y \le 2$; $z = 0 \Rightarrow 0 = 4 - y^2$
$\Rightarrow y = \pm 2$; $\mathbf{r}_x = \mathbf{i}$ and $\mathbf{r}_y = \mathbf{j} - 2y\mathbf{k} \Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & -2y \end{vmatrix} = 2y\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma$
$= \mathbf{F}\cdot\frac{\mathbf{r}_x\times\mathbf{r}_y}{|\mathbf{r}_x\times\mathbf{r}_y|}|\mathbf{r}_x\times\mathbf{r}_y|\,dydx = (2xy - 3z)\,dydx = [2xy - 3(4-y^2)]\,dydx \Rightarrow \iint\limits_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$
$= \int_0^1 \int_{-2}^2 (2xy + 3y^2 - 12)\,dydx = \int_0^1 \left[xy^2 + y^3 - 12y\right]_{-2}^2 dx = \int_0^1 -32dx = -32$
2. Let the parametrization be $\mathbf{r}(x,y) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$, $-1 \le x \le 1$, $0 \le z \le 2 \Rightarrow \mathbf{r}_x = \mathbf{i} + 2x\mathbf{j}$ and $\mathbf{r}_z = \mathbf{k}$
$\Rightarrow \mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2x\mathbf{i} - \mathbf{j} \Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\frac{\mathbf{r}_x\times\mathbf{r}_z}{|\mathbf{r}_x\times\mathbf{r}_z|}|\mathbf{r}_x\times\mathbf{r}_z|\,dzdx = -x^2\,dzdx$
$\Rightarrow \iint\limits_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_{-1}^1 \int_0^2 -x^2\,dzdx = -\frac{4}{3}$

Exercise 21.

In the following exercises, use a parametrization to find the flux

$$\iint_S \mathrm{F} \cdot \mathrm{n}\, d\sigma$$

across the surface in the given direction.

1. Sphere : $\mathrm{F} = z\mathrm{k}$ *across the portion of the sphere* $x^2 + y^2 + z^2 = a^2$ *in the first octant in the direction away from the origin*
2. Cylinder : $\mathrm{F} = x\mathrm{i} + y\mathrm{j} + z\mathrm{k}$ *outward through the portion of the cylinder* $x^2 + y^2 = 1$ *cut by the planes* $z = 0$ *and* $z = a$

Solution for Exercise 21

1. Let the parametrization be $\mathbf{r}(\phi,\theta) = (a\sin\phi\cos\theta)\mathbf{i} + (a\sin\phi\sin\theta)\mathbf{j} + (a\cos\phi)\mathbf{k}$ (spherical coordinates with $\rho = a, a \geq 0$, on the sphere),
   $0 \leq \phi \leq \frac{\pi}{2}$ (for the first octant), $0 \leq \theta \leq \frac{\pi}{2}$ (for the first octant)
   $\Rightarrow \mathbf{r}_\phi = (a\cos\phi\cos\theta)\mathbf{i} + (a\cos\phi\sin\theta)\mathbf{j} - (a\sin\phi)\mathbf{k}$ and
   $\mathbf{r}_\theta = (-a\sin\phi\sin\theta)\mathbf{i} + (a\sin\phi\cos\theta)\mathbf{j}$
   $$\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$
   $= (a^2\sin^2\phi\cos\theta)\mathbf{i} + (a^2\sin^2\phi\sin\theta)\mathbf{j} + (a^2\sin\phi\cos\phi)\mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\frac{\mathbf{r}_\phi\times\mathbf{r}_\theta}{|\mathbf{r}_\phi\times\mathbf{r}_\theta|}|\mathbf{r}_\phi\times\mathbf{r}_\theta|\,d\theta\,d\phi$
   $= a^3\cos^2\phi\sin\phi\,d\theta\,d\phi$ since
   $\mathbf{F} = z\mathbf{k} = (a\cos\phi)\mathbf{k} \Rightarrow \iint\limits_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{\pi/2}\int_0^{\pi/2} a^3\cos^2\phi\sin\phi\,d\phi\,d\theta = \frac{\pi a^3}{6}$

2. Let the parametrization be $\mathbf{r}(\theta,z) = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + z\mathbf{k}$, $0 \leq z \leq a$, $0 \leq \theta \leq 2\pi$ (where $r = \sqrt{x^2+y^2} = 1$ on the cylinder) $\Rightarrow \mathbf{r}_\theta = (-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$ and
   $$\mathbf{r}_z = \mathbf{k} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$$
   $\Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\frac{\mathbf{r}_\theta\times\mathbf{r}_z}{|\mathbf{r}_\theta\times\mathbf{r}_z|}|\mathbf{r}_\theta\times\mathbf{r}_z|\,dz\,d\theta = (\cos^2\theta+\sin^2\theta)\,dz\,d\theta = dz\,d\theta$, since
   $\mathbf{F} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + z\mathbf{k}$
   $\Rightarrow \iint\limits_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_0^a 1\,dz\,d\theta = 2\pi a$

Exercise 22.

In the following exercises, use a parametrization to find the flux

$$\iint_S \mathrm{F} \cdot \mathrm{n}\, d\sigma$$

across the surface in the given direction.

1. Cone : $\mathrm{F} = y^2\mathrm{i} + xz\mathrm{j} - \mathrm{k}$ *outward (normal away from the z-axis) through the cone* $z = 2\sqrt{x^2 + y^2}$, $0 \leq z \leq 2$
2. Paraboloid : $\mathrm{F} = 4x\mathrm{i} + 4y\mathrm{j} + 2\mathrm{k}$ *outward (normal away from the z-axis) through the surface cut from the bottom of the paraboloid* $z = x^2 + y^2$ *by the plane* $z = 1$

Solution for Exercise 22

1. Let the parametrization be $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + 2r\mathbf{k}$, $0 \le r \le 1$ (since $0 \le z \le 2$) and $0 \le \theta \le 2\pi$
$\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 2\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_r =$
$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 2 \end{vmatrix} = (2r\cos\theta)\mathbf{i} + (2r\sin\theta)\mathbf{j} - r\mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\frac{\mathbf{r}_\theta\times\mathbf{r}_r}{|\mathbf{r}_\theta\times\mathbf{r}_r|}|\mathbf{r}_\theta\times\mathbf{r}_r|\,d\theta\,dr$
$= (2r^3\sin^2\theta\cos\theta + 4r^3\cos\theta\sin\theta + r)\,d\theta\,dr$ since $\mathbf{F} = (r^2\sin^2\theta)\mathbf{i} + (2r^2\cos\theta)\mathbf{j} - \mathbf{k}$
$\Rightarrow \iint\limits_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_0^1 (2r^3 sin^2\theta\cos\theta + 4r^3\cos\theta\sin\theta + r)\,dr\,d\theta =$
$\int_0^{2\pi}(\frac{1}{2}\sin^2\theta\cos\theta + \cos\theta\sin\theta + \frac{1}{2})\,d\theta = \left[\frac{1}{6}\sin^3\theta + \frac{1}{2}\sin^2\theta + \frac{1}{2}\theta\right]_0^{2\pi} = \pi$
2. Let the parametrization be $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r^2\mathbf{k}$, $0 \le r \le 1$ (since $0 \le z \le 1$) and $0 \le \theta \le 2\pi$
$\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 2r\mathbf{k}$ and
$\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 2r \end{vmatrix} =$
$(2r^2\cos\theta)\mathbf{i} + (2r^2\sin\theta)\mathbf{j} - r\mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\frac{\mathbf{r}_\theta\times\mathbf{r}_r}{|\mathbf{r}_\theta\times\mathbf{r}_r|}|\mathbf{r}_\theta\times\mathbf{r}_r|\,d\theta\,dr$
$= (8r^3\cos^2\theta + 8r^3\sin^2\theta - 2r)\,d\theta\,dr = (8r^3 - 2r)\,d\theta\,dr$ since
$\mathbf{F} = (4r\cos\theta)\mathbf{i} + (4r\sin\theta)\mathbf{j} + 2\mathbf{k} \Rightarrow \iint\limits_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_0^1 (8r^3 - 2r)\,dr\,d\theta = 2\pi$

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